Exercises

Determinants – Solutions

Exercise 1.

(a)

$$\det\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 1\cdot 4 - 2\cdot 2 = 0$$

$$\det\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = 1\cdot 3 - 2\cdot 2 = -1$$

$$\det\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} = 14 - 16 = -2$$

(b)

$$\det\begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -2 \\ 1 & 4 & 2 \end{pmatrix} = 4 - 6 + 0 - (-1) - (-16) - 0 = 15$$

$$\det\begin{pmatrix} 5 & 5 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} = 5 + 10 + 1 - 2 - 5 - 5 = 4$$

(c)

$$\begin{aligned}\det\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} &= (a+b)^2 - (a-b)^2 \\ &= a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) = 4ab\end{aligned}$$

Exercise 2.

(a) Expansion along the first column yields

$$\begin{aligned}\det(A_1) &= \underbrace{a_{11}}_{1}\, d_{11} + \underbrace{a_{21}}_{0}\, d_{21} + \underbrace{a_{31}}_{0}\, d_{31} + \underbrace{a_{41}}_{0}\, d_{41} \\ &= a_{11}\, d_{11} \\ &= 1\cdot d_{11} = (-1)^{1+1}\, |A_{11}| \\ &= \begin{vmatrix} 2 & 3 & -1 \\ 0 & 1 & -2 \\ 1 & 4 & 2 \end{vmatrix} = 15\end{aligned}$$

(b) Expansion along the fourth row yields

$$\begin{aligned}\det(A_2) &= \underbrace{a_{41}}_{0}\ d_{41} + \underbrace{a_{42}}_{0}\ d_{42} + \underbrace{a_{43}}_{0}\ d_{43} + \underbrace{a_{44}}_{2}\ d_{44} \\ &= a_{44}\ d_{44} \\ &= 2 \cdot d_{44} \ = \ 2\,(-1)^{4+4}\,|A_{44}| \\ &= 2 \cdot \begin{vmatrix} 5 & 5 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} \ = \ 2 \cdot 4 \ = \ 8\end{aligned}$$

Exercise 3. One possible counter example is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\det(A) = \det(B) = 0$ (i.e. $\det(A) + \det(B) = 0$) but

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and thus $\det(A + B) = 1 \neq 0$.

Exercise 4.

(a) We have

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & a & 3 \\ 1 & 2 & a \end{pmatrix} \ = \ a^2{+}6{+}6{-}3a{-}6{-}2a \ = \ a^2{-}5a{+}6 \ = \ (a{-}2)(a{-}3)$$

Thus

$$\det(A) \ = \ 0 \iff (a-2)(a-3) \ = \ 0 \iff \underline{\underline{a = 2 \text{ or } a = 3}}.$$

$$\det(A) \ > \ 0 \iff (a-2)(a-3) \ > \ 0 \iff \underline{\underline{a > 3 \text{ or } a < 2}}.$$

$$\det(A) \ < \ 0 \iff \underline{\underline{a \ \in\,]2,\,3[}}.$$

(b) We have

$$\det \begin{pmatrix} b & 1 & 2 \\ -1 & b & 0 \\ 2 & 0 & -1 \end{pmatrix} \ = \ -b^2 - 4b - 1 \ \stackrel{!}{=} \ 0$$

$$\begin{aligned} \Leftrightarrow \quad -b^2 - 4b - 1 &= 0 \\ b^2 + 4b + 1 &= 0 \\ b_{1,2} &= -2 \pm \sqrt{4-1} \end{aligned}$$

For $\underline{\underline{b_1 = -2 + \sqrt{3}}}$ or $\underline{\underline{b_2 = -2 - \sqrt{3}}}$ we have $\det(B) = 0$.

(c) We have

$$\det(B) = f(b) = -b^2 - 4b - 1 = \underbrace{-(b+2)^2}_{\leq 0} + 3 \leq 3.$$

So $\det(B) = f(b)$ is maximal for $b = -2$ with $f(-2) = 3$.

Exercise 5.

1. (a) We compute

$$\det A = \det \begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix} = -8$$

$$\det A_1 = \det \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix} = -4 \qquad \det A_2 = \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = -1$$

and thus get

$$x_1 = \frac{\det A_1}{\det A} = \frac{-4}{-8} = \frac{1}{2} \qquad x_2 = \frac{\det A_2}{\det A} = \frac{-1}{-8} = \frac{1}{8}$$

$$\Leftarrow \quad x = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{8} \end{pmatrix}$$

(b) We compute

$$\det A = \det \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} = 6$$

$$\det A_1 = \det \begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix} = 10 \qquad \det A_2 = \det \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} = 2$$

and thus get

$$x_1 = \frac{\det A_1}{\det A} = \frac{10}{6} = \frac{5}{3} \qquad x_2 = \frac{\det A_2}{\det A} = \frac{2}{6} = \frac{1}{3}$$

$$\Leftarrow \quad x = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \end{pmatrix}$$

2. We compute

$$\det A = \det \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ -1 & 1 & 0 \end{pmatrix} = 6$$

$$\det A_1 = \det \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ -4 & 1 & 0 \end{pmatrix} = 18$$

$$\det A_2 = \det \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ -1 & -2 & 0 \end{pmatrix} = -6$$

$$\det A_1 = \det \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -4 \end{pmatrix} = 12$$

and thus get

$$x_1 = \frac{\det A_1}{\det A} = \frac{18}{6} = 3 \quad x_2 = \frac{\det A_2}{\det A} = \frac{-6}{6} = -1 \quad x_3 = \frac{\det A_3}{\det A} = \frac{12}{6} = 2$$

$$\Leftarrow \quad x = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$